Assignment 5

Coverage: 15.7 in Text.

Exercises: 15.7 no 10, 12, 14, 16, 18, 20, 26, 32, 38, 42, 54, 62, 66.

Submit 15.7 no. 12, 16, 66 by Oct 12.

Supplementary Exercises

1. Find the equation of the plane passing through the following three points: (a) $(0,0,0),(1,2,3),(0,1,-1)$, (b) $(0,0,0),(4,2,0),(-9,0,1)$, (c) $(-1,1,0),(0,7,0),(2,0,1)$.
2. Write down the equations of the planes constituting the tetrahedron with vertices at $(0,0,0),(2,0,0),(0,1,0)$ and $(0,0,3)$.
3. Write down the equations of the planes constituting the tetrahedron with vertices at $(0,0,0),(2,0,0),(0,2,0),(1,1,5)$.

Equation of a Plane in Space

A tetrahedron is made of the intersection of four planes in space. It can also be determined by its four vertices. In practise we need to write down the equation of its constituting planes from its vertices. This leads to the determination of the equation of a plane when three points on it are given.

Recall that the equation of a plane in space is given by $ax+by+cz=d$, that is, the plane is the set

$$\{(x,y,z):\ ax+by+cz=d\}\ .$$

The plane passes through the origin if and only if $d=0$. In this case, $ax+by+cz=0$ can be written in dot product $(a,b,c)\cdot(x,y,z)=0$, so the vector $\mathbf{n}\equiv(a,b,c)$ is perpendicular to all points (or vectors) (x,y,z) on the plane. In other words, $\mathbf{n}$ is normal to the plane. Now, when two vectors $\mathbf{u}$ and $\mathbf{v}$ on the plane are given, we may use the relations $\mathbf{n}\cdot\mathbf{u}=0$ and $\mathbf{n}\cdot\mathbf{v}=0$ to determine $\mathbf{n}$. In fact, $\mathbf{n}$ could be chosen to be the cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$. Note that $\mathbf{n}$ is not unique; any non-zero multiple of $\mathbf{n}$ is again perpendicular to the plane.

We have the following formula for the cross product

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}\ .$$

Example 1. Find the equation of the plane passing through $(0,0,0),(1,2,0)$, and $(1,0,-1)$.

The normal direction could be chosen to be $(1,2,0)\times(1,0,-1)$. Using the formula above, we have $\mathbf{n}=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$. So the equation of this plane is $(-2,1,-2)\cdot(x,y,z)=0$, or $-2x+y-2z=0$.

When $d\neq 0$, we first translate the plane to the plane containing the origin. After determining (a,b,c) we determine d. Here is an example.

Example 2. Find the equation of the plane passing through $(1,0,0),(2,0,-1),(0,1,0)$. By subtracting these three vectors from the first one we get $(0,0,0),(1,0,-1),(-1,1,0)$. Now these three points form a plane passing through the origin. Using the formula above, its normal direction is given by $(1,0,-1)\times(-1,1,0)=(1,1,1)$. So the plane is $x+y+z=0$. Our plane is of the form $x+y+z=d$. To determine d, note $(1,0,0)$ lies on the plane, so $1+0+0=d$, that is, $d=1$. We conclude the equation for the plane is $x+y+z=1$.

Example 3. Let T be the tetrahedron with vertices $(0,0,0),(1,1,0),(0,1,0)$ and $(0,1,1)$. Find the equations of its four constituting planes. By sketching the figures, we see that three planes are simple ones: $z=0$ (the xy-plane), $x=0$ (the yz-plane), and $y=1$ (the plane passing through $(0,1,0)$ and parallel to the xz-plane. The fourth plane is the one passing through $(0,0,0),(1,1,0)$ and $(0,1,1)$. From $(1,1,0)\times(0,1,1)=(1,-1,1)$ we see that this plane is given by $x-y+z=0$.